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# UNIQUENESS AND BIFURCATION IN ELASTIC–PLASTIC SOLIDS

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**Abstract**—The present work is concerned with the problems of uniqueness and bifurcation in a time-independent elastic–plastic material obeying the normality flow rule with a smooth yield surface. Rate $n$ constitutive relations are formulated for the corresponding material. Under self-adjoint boundary conditions, it is shown that every solution of the rate $n$ boundary value problem is governed by a variational principle and the corresponding functional reaches a strict absolute minimum if the solution satisfies a sufficient uniqueness condition. In order to analyse uniqueness and bifurcation, a series of linear comparison solids are constructed. It is shown that Hill's exclusion condition excludes not only rate one bifurcations but also higher order ones.

## 1. INTRODUCTION

The problems of uniqueness and bifurcation in elastic–plastic solids are important in practice and difficult in mathematics. Over a long period of time, the problems have been under intensive investigations. Following some early work, Hill (1958, 1959) investigated the uniqueness in velocities and the corresponding bifurcation for the first order rate boundary value problems of elastic–plastic solids. He postulated that no bifurcation would occur at a load if the first order rate boundary value problem has a unique solution. But, Hill (1978) pointed out that the nonuniqueness is probably always reducible to that by reference to rate problems of higher order.

The issue of higher order bifurcation was first addressed by Triantafyllidis (1983), but his approach is limited to a particular class of constitutive laws developed by Christoffersen and Hutchinson (1979). The second order rate boundary value problem was posed by Petryk and Thermann (1985) for elastic–plastic solids with prescribed nominal tractions and displacement functions of time on the respective parts of the body surface. In their work, it is shown that not only the solution of the first order rate but also the solution of the second order rate is unique if Hill's sufficient uniqueness condition is satisfied. Some conditions were indicated under which a second order bifurcation becomes possible. The theory of any higher order bifurcation was proposed by Nguyen and Triantafyllidis (1989), via the use of the generalized standard material formalism, but their theory is applicable only to the case of small strain.

The aim of the present work is to extend the uniqueness and bifurcation theories of elastic–plastic solids mentioned above. First, rate $n$ constitutive equations are formulated for a time-independent elastic–plastic material obeying the normality flow rule with a smooth yield surface. Second, rate $n$ boundary value problems are posed, and the corresponding variational principles are established if the boundary conditions are self-adjoint. Finally, the uniqueness and bifurcation are discussed. A series of linear comparison solids are constructed. It is shown that Hill's uniqueness condition excludes not only first order bifurcation but also bifurcations of any higher order.

## 2. CONSTITUTIVE RELATIONS

For the material considered in this paper, according to Hill (1978), we make the following assumption. In the strain space, there is a domain within which the response is purely elastic. The boundary of the domain is called yield surface. If the incremental deformation path leaves the fixed domain which coincides momentarily with the elastic domain, the incremental response is nonelastic. Further, we assume that the yield surfaces are sufficiently smooth and the normality flow rule holds.

The stress and strain, designated by **T** and **E** respectively, are objective measures, and **T** is work-conjugate to **E** in the sense of Hill (1968, 1978). With **H** symbolizing the history of inelastic straining, the equation of a yield surface can be written as

$$F(\mathbf{E}, \mathbf{H}) = 0, \tag{1}$$

where $F$ is a sufficiently smooth scalar function, with a non-vanishing gradient $\partial F/\partial \mathbf{E}$ on the yield surface, and $F < 0$ in the corresponding elastic domain. Define

$$\Omega = \{\mathbf{E} | F(\mathbf{E}, \mathbf{H}) < 0\}, \tag{2}$$

$$\partial\Omega = \{\mathbf{E} | F(\mathbf{E}, \mathbf{H}) = 0\}, \tag{3}$$

$$\mathbf{\Lambda} = \frac{\partial F}{\partial \mathbf{E}}. \tag{4}$$

The constitutive rate equation can be written in the form (Hill, 1978)

$$\dot{\mathbf{T}} = \mathbf{L}\dot{\mathbf{E}}, \tag{5}$$

$$\mathbf{L} = \mathscr{L} - \frac{\alpha}{g} \mathbf{\Lambda} \otimes \mathbf{\Lambda}. \tag{6}$$

In (5) the dots denote the material time derivatives. In (6), $g$ is a positive scalar parameter, $\mathscr{L}$ is the positive definite tensor of instantaneous elastic moduli with symmetries $\mathscr{L}_{ijkl} = \mathscr{L}_{klij} = \mathscr{L}_{jikl}$, and the value of $\alpha$ is determined as follows

$$\begin{aligned} &\alpha = 0, \quad \text{if } \mathbf{E} \in \Omega \text{ or } \mathbf{E} \in \partial\Omega \text{ and } \mathbf{\Lambda} \cdot \dot{\mathbf{E}} < 0, \\ &\alpha = 1, \quad \text{if } \mathbf{E} \in \partial\Omega \text{ and } \mathbf{\Lambda} \cdot \dot{\mathbf{E}} > 0. \end{aligned} \tag{7}$$

The criterion (7) is sufficient for the first-order rate problem, but not for higher order rate problems. In fact, if $\mathbf{E} \in \partial\Omega$, $\mathbf{\Lambda} \cdot \dot{\mathbf{E}} = 0$ and only the first order rate problem is concerned, we have no need to define $\alpha$ [cf. Petryk and Thermann (1985)]. In order to establish constitutive equations of rate $n \geqslant 2$, we need to expand $F(\mathbf{E} + \delta\mathbf{E}, \mathbf{H})$ into a series. However, if $F(\mathbf{E} + \delta\mathbf{E}, \mathbf{H})$ is expanded directly into a series in terms of $\delta\mathbf{E}$, the later discussion will be cumbersome. Strain **E** is a function of time $t$ and $\mathbf{E}(t) + \delta\mathbf{E} = \mathbf{E}(t + \delta t)$, so $F(\mathbf{E} + \delta\mathbf{E}, \mathbf{H})$ can be expanded into a series in terms of $\delta t$. In the following computation, we fix **H** and make use of the relation

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} = \mathbf{\Lambda} \cdot \dot{\mathbf{E}}. \tag{8}$$

Then we have

$$F(\mathbf{E}+\delta\mathbf{E},\mathbf{H})-F(\mathbf{E},\mathbf{H}) = \sum_{i=1}^{\infty}\frac{1}{i!}\frac{\mathrm{d}^iF}{\mathrm{d}t^i}\bigg|_{(\mathbf{E},\mathbf{H})}(\delta t)^i$$

$$= (\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}})|_{(\mathbf{E},\mathbf{H})}\delta t+\sum_{i=1}^{\infty}\frac{1}{(i+1)!}\frac{\mathrm{d}^i}{\mathrm{d}t^i}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}})|_{(\mathbf{E},\mathbf{H})}(\delta t)^{i+1}. \tag{9}$$

If $\mathbf{E}\in\partial\Omega$ and $\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}} = 0$, the first-order rate of plastic straining (viz. $\dot{\mathbf{H}}$) vanishes. So when computing the second term in (9), we have no need to fix $\mathbf{H}$. In general, if $\mathbf{E}\in\partial\Omega$, $\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}} = 0$ and

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}}) = 0, \quad \text{for } k = 1,2,\ldots,n-2, \tag{10}$$

we make the assumption (this will be verified *a posteriori*)

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{H} = 0, \quad \text{for } k = 1,2,\ldots,n-1. \tag{11}$$

If (11) holds, in the computation of the $n$th term in the series (9), $\mathbf{H}$ needn't be fixed.

In view of the assumption made at the beginning of this section, a general criterion to determine $\alpha$ is

$$\begin{aligned}\alpha = 0, &\quad \text{if } \mathbf{E}\in\Omega \text{ or } \mathbf{E}\in\partial\Omega \text{ and } F(\mathbf{E}+\delta\mathbf{E},\mathbf{H}) < 0,\\ \alpha = 1, &\quad \text{if } \mathbf{E}\in\partial\Omega \text{ and } F(\mathbf{E}+\delta\mathbf{E},\mathbf{H}) > 0,\end{aligned} \tag{12}$$

where $\delta\mathbf{E}$ is the sufficiently small, non-zero strain increment taken along the actual deformation path in the sufficiently small time increment $\delta t$.

The criterion (12) and expression (9) with the assumption (11) lead us to the following loading/unloading criterion of order $n \geqslant 2$.

When $\mathbf{E}\in\Omega$, $\alpha = 0$. When $\mathbf{E}\in\partial\Omega$,

$\alpha = 0$, if $\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}} < 0$ or $\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}} = 0$ and $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}}) < 0$ for the smallest natural number $k \leqslant n-1$ which makes $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}}) \neq 0$;

$\alpha = 1$, if $\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}} > 0$ or $\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}} = 0$ and $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}}) > 0$ for the smallest natural number $k \leqslant n-1$ which makes $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}}) \neq 0$. (13)

Differentiating (5) $n-1$ times with respect to time-like parameter results in the rate $n$ constitutive equation

$$\frac{\mathrm{d}^n\mathbf{T}}{\mathrm{d}t^n} = \mathbf{L}\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}+(n-1)\dot{\mathbf{L}}\frac{\mathrm{d}^{n-1}\mathbf{E}}{\mathrm{d}t^{n-1}}+\cdots+\frac{\mathrm{d}^{n-1}\mathbf{L}}{\mathrm{d}t^{n-1}}\dot{\mathbf{E}} \tag{14a}$$

or

$$\frac{\mathrm{d}^n\mathbf{T}}{\mathrm{d}t^n} = \mathscr{L}\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}+(n-1)\dot{\mathscr{L}}\frac{\mathrm{d}^{n-1}\mathbf{E}}{\mathrm{d}t^{n-1}}+\cdots+\frac{\mathrm{d}^{n-1}\mathscr{L}}{\mathrm{d}t^{n-1}}\dot{\mathbf{E}}-\alpha\left\{\frac{\boldsymbol{\Lambda}}{g}\frac{\mathrm{d}^{n-1}\mathbf{E}}{\mathrm{d}t^{n-1}}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}})\right.$$

$$\left.+(n-1)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\Lambda}}{g}\right)\frac{\mathrm{d}^{n-2}\mathbf{E}}{\mathrm{d}t^{n-2}}(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}})+\cdots+\frac{\mathrm{d}^{n-1}\mathbf{E}}{\mathrm{d}t^{n-1}}\left(\frac{\boldsymbol{\Lambda}}{g}\right)(\boldsymbol{\Lambda}\cdot\dot{\mathbf{E}})\right\}, \tag{14b}$$

where $\alpha$ is determined by (7) for $n = 1$, and by (13) for $n \geqslant 2$. Let $n = 2$, then (14) reduces to the second order constitutive equation presented by Petryk and Thermann (1985). Obviously, the relation (14a) or (14b) between $n$th order rates of **T** and **E** is piecewise-linear and continuous.

If $\mathbf{E} \in \partial\Omega$, $\mathbf{\Lambda}\cdot\dot{\mathbf{E}} = 0$ and $\mathrm{d}^k(\mathbf{\Lambda}\cdot\dot{\mathbf{E}})/\mathrm{d}t^k = 0$ for $k = 1, 2, \ldots, n-1$, the value of $\alpha$ is not defined by the criterion (13). But, for this case, the constitutive relation (14b) can be simplified into the elastic constitutive relation

$$\frac{\mathrm{d}^n\mathbf{T}}{\mathrm{d}t^n} = \mathscr{L}\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n} + (n-1)\dot{\mathscr{L}}\frac{\mathrm{d}^{n-1}\mathbf{E}}{\mathrm{d}t^{n-1}} + \cdots + \frac{\mathrm{d}^{n-1}\mathscr{L}}{\mathrm{d}t^{n-1}}\dot{\mathbf{E}}.$$

So the $n$th-order rate of plastic straining (viz. $\mathrm{d}^n\mathbf{H}/\mathrm{d}t^n$) must be zero. Now, it is obvious that the assumption (11) is correct.

As an example, for the simplest flow theory of plasticity, $J_2$ flow theory, $F$ and $\mathbf{\Lambda}$ can be explicitly written

$$F = G(\mathbf{E}-\mathbf{E}^{\mathrm{p}}-\tfrac{1}{3}\mathbf{I}\otimes\mathbf{IE})\cdot(\mathbf{E}-\mathbf{E}^{\mathrm{p}})-k,$$

$$\mathbf{\Lambda} = \frac{\partial F}{\partial \mathbf{E}} = 2G(\mathbf{E}-\mathbf{E}^{\mathrm{p}}-\tfrac{1}{3}\mathbf{I}\otimes\mathbf{IE}),$$

where $G$ is the elastic shear modulus, **I** is the metric tensor, $\mathbf{E}^{\mathrm{p}}$ is plastic strain, $k$ and $g$ are functions of $\mathbf{E}^{\mathrm{p}}\cdot\mathbf{E}^{\mathrm{p}}$. Here **H** is replaced by $\mathbf{E}^{\mathrm{p}}$. Since a material element deforms elastically within the elastic domain, $\mathbf{\Lambda}$ equals stress deviator. If the relation between uniaxial stress and strain and solutions of rate one to rate $n-1$ are known, the rate $n$ constitutive equation can be written out explicitly.

For the sake of later considerations, the constitutive relations will be reformulated in terms of nominal stress **N** and deformation gradient **A**. Henceforth, we identify **T** and **E** with the second Piola–Kirchhoff stress tensor and the Green strain tensor. The constitutive equations (5) can be transformed into the following form:

$$\dot{\mathbf{N}} = \mathbf{C}\dot{\mathbf{A}}^{\mathrm{T}}, \tag{15}$$

where $\dot{\mathbf{A}}^{\mathrm{T}}$ is the transpose of $\dot{\mathbf{A}}$.

On a fixed Cartesian coordinate frame, the components of **C** can be expressed in terms of the components of **L** (Hill, 1978).

$$\begin{aligned} C_{ijkl} &= A_{jp}A_{lq}L_{ipkq} + T_{ik}\delta_{jl} \\ &= A_{jp}A_{lq}\mathscr{L}_{ipkq} + T_{ik}\delta_{jl} - \frac{\alpha}{g}A_{jp}A_{lq}\Lambda_{ip}\Lambda_{kq}. \end{aligned} \tag{16}$$

Since **E** is a function of **A**, we can define a tensor $\boldsymbol{\eta}$ by the following relation:

$$\boldsymbol{\eta}^{\mathrm{T}} = \frac{\partial F}{\partial \mathbf{A}} = \mathbf{A\Lambda}. \tag{17}$$

Introducing another tensor $\mathscr{C}$ whose components are

$$\mathscr{C}_{ijkl} = A_{jp}A_{lq}\mathscr{L}_{ipkq} + T_{ik}\delta_{jl} \tag{18}$$

then we can write **C** in the form

$$\mathbf{C} = \mathscr{C} - \frac{\alpha}{g}\boldsymbol{\eta}\otimes\boldsymbol{\eta}. \tag{19}$$

Corresponding to sets (2) and (3) are the sets

$$\omega = \{\mathbf{A}|F(\mathbf{E}(\mathbf{A}),\mathbf{H}) < 0\}, \tag{20}$$

$$\partial\omega = \{\mathbf{A}|F(\mathbf{E}(\mathbf{A}),\mathbf{H}) = 0\}. \tag{21}$$

By the definition of Green strain, we have

$$\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}\cdot\dot{\mathbf{E}}. \tag{22}$$

With having the above notations and identity (22), the criterion (7) and (13) can be rewritten as follows:

For $\mathbf{A}\in\omega$, $\alpha = 0$. For $\mathbf{A}\in\partial\omega$,

when $n = 1$, $\alpha = 0$, *if* $\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} < 0$; $\alpha = 1$, if $\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} > 0$;

when $n \geqslant 2$,

$\alpha = 0$, if $\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} < 0$ or $\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} = 0$ and $\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}}) < 0$ for the

smallest natural number $k \leqslant n-1$ which makes $\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}}) \neq 0$;

$\alpha = 1$, if $\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} > 0$ or $\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}} = 0$ and $\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}}) > 0$ for the

smallest natural number $k \leqslant n-1$ which makes $\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}}) \neq 0$. (23)

The constitutive equations (14a, b) are transformed into the equations

$$\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n} = \mathbf{C}\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} + (n-1)\dot{\mathbf{C}}\frac{\mathrm{d}^{n-1}\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^{n-1}} + \cdots + \frac{\mathrm{d}^{n-1}\mathbf{C}}{\mathrm{d}t^{n-1}}\dot{\mathbf{A}}^{\mathrm{T}} \tag{24a}$$

and

$$\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n} = \mathscr{C}\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} + (n-1)\dot{\mathscr{C}}\frac{\mathrm{d}^{n-1}\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^{n-1}} + \cdots + \frac{\mathrm{d}^{n-1}\mathscr{C}}{\mathrm{d}t^{n-1}}\dot{\mathbf{A}}^{\mathrm{T}} - \alpha\left\{\frac{\boldsymbol{\eta}}{g}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}}) + (n-1)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\eta}}{g}\right)\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}}) + \cdots + \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left(\frac{\boldsymbol{\eta}}{g}\right)(\boldsymbol{\eta}\cdot\dot{\mathbf{A}}^{\mathrm{T}})\right\}. \tag{24b}$$

Though nominal stress is not an objective measure, constitutive equations (24a, b) have the same form as (14a, b). But $\mathbf{C}$ is not a true moduli tensor since it is partly coupled to the material spin.

## 3. ANY ORDER RATE PROBLEM

The field equations will be written from a Lagrangian standpoint, in terms of nominal stress. A volume element in the reference configuration is denoted by $\mathrm{d}\tau$, a surface element by $\mathrm{d}s$. $\gamma$ is the body force per unit reference volume. For any part of the body, the equilibrium equation is

$$\int \mathbf{N}^{\mathrm{T}}\,\mathrm{d}\mathbf{s}+\int \gamma\,\mathrm{d}\tau = 0. \tag{25}$$

Here we restrict ourselves to considering the problems in which $\mathrm{d}^k\mathbf{N}/\mathrm{d}t^k$ is a piecewise-smooth and continuous function of spatial coordinates for $k < n$. By differentiating (25) $n$ times with respect to time, we obtain the rate $n$ equilibrium equation:

$$\int \frac{\mathrm{d}^n\mathbf{N}^{\mathrm{T}}}{\mathrm{d}t^n}\,\mathrm{d}\mathbf{s}+\int \frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n}\,\mathrm{d}\tau = 0. \tag{26}$$

If the $n$th rate of $\mathbf{N}$ is continuously differentiable with respect to spatial coordinates, by the divergence theorem, eqn (26) implies

$$\operatorname{div}\left(\frac{\mathrm{d}^n\mathbf{N}^{\mathrm{T}}}{\mathrm{d}t^n}\right)+\frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n} = 0, \tag{27}$$

where div designates the material divergence.

If there is a surface $S_{\mathrm{D}}$ in the body, across which the $n$th rate of $\mathbf{N}$ is discontinuous, from the global equation (26) it can be deduced that the $n$th rate of nominal traction is continuous. With $[\![\cdot]\!]$ denoting the jump, this can be expressed in the form

$$\boldsymbol{\nu}\left[\!\!\left[\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right]\!\!\right] = 0 \quad \text{on } S_{\mathrm{D}}, \tag{28}$$

where $\boldsymbol{\nu}$ denotes the unit vector normal to the surface $S_{\mathrm{D}}$.

To encompass a wide class of configuration-dependent loading in Hill (1962), it is stipulated that the nominal traction rate is a linear function of the particle velocity and its gradient:

$$\dot{\mathbf{N}}^{\mathrm{T}}\,\mathrm{d}\mathbf{s} = (\dot{\mathbf{d}}+\mathbf{f}(\dot{\mathbf{x}}))\,\mathrm{d}s, \tag{29}$$

where $\dot{\mathbf{b}}$ is a vector (independent of material response), $\mathbf{f}$ is a linear homogeneous expression of the velocity $\dot{\mathbf{x}}$ and its gradient, $\mathbf{x}$ the position vector of a material point in the current configuration, $\mathrm{d}s$ the area of the surface element $\mathrm{d}\mathbf{s}$. Differentiating (29) $n-1$ times with respect to time, we have

$$\frac{\mathrm{d}^n\mathbf{N}^{\mathrm{T}}}{\mathrm{d}t^n}\,\mathrm{d}\mathbf{s} = \left(\frac{\mathrm{d}^n\mathbf{b}}{\mathrm{d}t^n}+\mathbf{f}\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)+\mathbf{R}_n\right)\mathrm{d}s. \tag{30}$$

In (30), $\mathbf{R}_n$ is a function of $\mathrm{d}^k\mathbf{x}/\mathrm{d}t^k$ $(k = 1,2,\ldots,n-1)$, and $\mathbf{R}_1 = 0$.

The boundary conditions are called homogeneous when $S_{\mathrm{u}}$ is fixed and $\mathrm{d}^n\mathbf{b}/\mathrm{d}t^n$ vanishes. Homogeneous conditions are self-adjoint (Hill, 1978) when

$$\int \mathbf{f}(\mathbf{v})\cdot\delta\mathbf{v}\,\mathrm{d}s = \tfrac{1}{2}\delta\int \mathbf{f}(\mathbf{v})\cdot\mathbf{v}\,\mathrm{d}s, \tag{31}$$

where $\mathbf{v}$ is an arbitrary, continuous and piecewise-continuously differentiable vector field, $\delta\mathbf{v}$ is any infinitesimal variation that vanishes on $S_{\mathrm{u}}$.

For the rate $n$ problem, we consider only the boundary conditions that on a part $S_{\mathrm{T}}$ of the body surface, the $n$th rate of $\mathbf{N}$ satisfies (30), and on the remaining part $S_{\mathrm{u}}, \mathbf{x}(t)$ is a given function of time. The rate $n$ problem consists of the constitutive equation (24a) or (24b), equilibrium equations (27) and (28), and the boundary conditions prescribed here.

When we begin to solve the $n$th problem we assume that the solutions of rate one to $n-1$ problems are known.

The first-order rate constitutive equation (15) can be rewritten in the form (cf. Hill, 1959)

$$\dot{\mathbf{N}} = \frac{\mathrm{d}U_1}{\partial \dot{\mathbf{A}}^{\mathrm{T}}}, \quad U_1 = \tfrac{1}{2}\dot{\mathbf{A}}^{\mathrm{T}} \cdot \mathbf{C}\dot{\mathbf{A}}^{\mathrm{T}}. \tag{32}$$

For the $n$th-order rate constitutive equation (24), we construct a function

$$U_n = -\frac{1}{2}\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} \cdot \mathbf{C}\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} + \frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} \cdot \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\mathbf{C}\dot{\mathbf{A}}^{\mathrm{T}}) - \frac{\alpha}{2g}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}}) - \boldsymbol{\eta} \cdot \frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right]^2. \tag{33}$$

Then we can write the $n$th-order rate constitutive equation in terms of $U_n$, i.e.

$$\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n} = \frac{\partial U_n}{\partial\left(\dfrac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)}. \tag{34}$$

In order to prove (34), we assume

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{A}}{\mathrm{d}t^n}\right) = \frac{\mathrm{d}^n\mathbf{A}}{\mathrm{d}t^n} - \frac{\mathrm{d}^n\mathbf{A}_0}{\mathrm{d}t^n}, \quad \Delta\left(\frac{\mathrm{d}^n\mathbf{A}}{\mathrm{d}t^n}\right) = 0, \quad k \leqslant n-1. \tag{35}$$

In view of the constitutive equation (24a) and the symmetry property of $\mathbf{C}$ ($C_{ijkl} = C_{klij}$), we have

$$\Delta U_n = \Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \cdot \frac{\mathrm{d}^n\mathbf{N}_0}{\mathrm{d}t^n} + \tfrac{1}{2}\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \cdot \mathbf{C}_0\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) + \frac{1}{2}\left[-\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} \cdot \Delta\mathbf{C}\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right.$$
$$\left. + 2\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n} \cdot \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\Delta\mathbf{C}\dot{\mathbf{A}}^{\mathrm{T}})\right] - \frac{\Delta\alpha}{2g}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}}) - \boldsymbol{\eta} \cdot \frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right]^2. \tag{36}$$

Since $\Delta\mathbf{C} = -\Delta\alpha\boldsymbol{\eta} \otimes \boldsymbol{\eta}/g$, the above expression reduces to the following:

$$\Delta U_n = \Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \cdot \frac{\mathrm{d}^n\mathbf{N}_0}{\mathrm{d}t^n} + \tfrac{1}{2}\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \cdot \mathbf{C}_0\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) - \frac{\Delta\alpha}{g}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}})\right]^2. \tag{37}$$

$\Delta\alpha$ does not vanish if and only if the following conditions are satisfied:

$$\mathbf{A} \in \partial\omega, \quad \boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}} = 0, \quad \frac{\mathrm{d}^k}{\mathrm{d}t^k}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}}) = 0, \quad \text{for } k = 1, 2, \ldots, n-2,$$
$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}_0^{\mathrm{T}})\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}}) \leqslant 0.$$

Hence,

$$|\Delta\alpha|\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\eta \cdot \dot{\mathbf{A}}^{\mathrm{T}})\right]^2 \leqslant |\Delta\alpha|\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}^{\mathrm{T}}) - \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\boldsymbol{\eta} \cdot \dot{\mathbf{A}}_0^{\mathrm{T}})\right]^2$$
$$= |\Delta\alpha|\left[\boldsymbol{\eta} \cdot \Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\right]^2. \tag{38}$$

From (37) and (38), it is easily seen that eqn (34) holds. If $n \geqslant 2$, then $U_n$ is not homogeneous in $\mathrm{d}^n\mathbf{A}/\mathrm{d}t^n$, while $U_1$ is necessarily homogeneous of degree two in velocity gradient.

Let displacements and their rate one to rate $n-1$ be smooth fields, $n$th rates of displacements be continuous and piecewise smooth. Let $\delta(\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n)$ be continuous and piecewise smooth and vanish over $S_u$. Further, we assume that $\delta(\mathrm{d}^k\mathbf{x}/\mathrm{d}t^k) = 0$ when $k = 1, 2, \ldots, n-1$. Then,

$$\int \frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n} \cdot \delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \mathrm{d}\tau = \int_{S_{\mathrm{T}}} \left(\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\,\delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\right) \cdot \mathrm{d}\mathbf{s} + \int \frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n} \cdot \delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) \mathrm{d}\tau \tag{39}$$

by (27), (28) and the divergence transformation. Making use of (34), we can rewrite (39) in the form

$$\delta \int \left(U_n - \frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n} \cdot \frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) \mathrm{d}\tau = \int_{S_{\mathrm{T}}} \left\{\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\,\delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\right\} \cdot \mathrm{d}\mathbf{s}. \tag{40}$$

When the condition on $S_{\mathrm{T}}$ is self-adjoint as in (30) with (31), then the following true variation principle is obtained:

$$\delta \prod_n \left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) = 0, \tag{41}$$

$$\prod_n = \int \left(U_n - \frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n} \cdot \frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) \mathrm{d}\tau - \int_{S_{\mathrm{T}}} \left(\frac{\mathrm{d}^n\mathbf{b}}{\mathrm{d}t^n} + \tfrac{1}{2}\mathbf{f}\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) + \mathbf{R}_n\right) \cdot \frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\,\mathrm{d}s. \tag{42}$$

Conversely, by reversing the analytical steps and applying the standard argument of the calculus of variations, it can be proved that some field $\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n$ is a solution of rate $n$ problem if it satisfies (41). For the first-order rate problem, variational principle (41) was given by Hill (1962).

## 4. UNIQUENESS

Suppose that the rate $n$ problem has two different solutions and the difference of corresponding quantities is denoted by the prefix $\Delta$. Suppose further that displacements, strains and stresses and their rate one to rate $n-1$ in the solutions discussed are the same. On the one hand, both solutions satisfy the equilibrium equations (27) and (28) and boundary conditions, on the other hand we have assumed that $\mathbf{f}$ is linear and $\mathbf{R}_n$ is independent of $\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n$ and its gradient, so we have

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) = 0 \quad \text{on } S_u, \tag{43}$$

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{N}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \mathrm{d}\mathbf{s} = \mathbf{f}\left(\Delta\,\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right) \quad \text{on } S_{\mathrm{T}}, \tag{44}$$

$$\mathrm{div}\left(\Delta\,\frac{\mathrm{d}^n\mathbf{N}^{\mathrm{T}}}{\mathrm{d}t^n}\right) = 0, \tag{45}$$

$$\mathbf{v}\left[\!\left[\Delta\left(\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right)\right]\!\right] = 0. \tag{46}$$

Hence, by some manipulation, we have

$$\int\Delta\left(\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\mathrm{d}\tau = \int\mathbf{f}\left(\Delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\mathrm{d}s. \tag{47}$$

A sufficient condition for uniqueness is therefore that

$$F_n = \int\Delta\left(\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\mathrm{d}\tau - \int\mathbf{f}\left(\Delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\mathrm{d}s > 0 \tag{48}$$

for all pairs of admissible fields taking the values prescribed on $S_{\mathrm{u}}$.

Now we turn to discuss the variational principle. Suppose that $\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n$ is a solution of rate $n$ problem and $\mathbf{w}$ is an arbitrary admissible field vanishing on $S_{\mathrm{u}}$. Write

$$\mathbf{a} = \frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n} + \lambda\mathbf{w}, \quad 0 \leqslant \lambda \leqslant 1.$$

Define

$$g(\lambda) = \prod_n(\mathbf{a}) - \prod_n\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right),$$

where $g$ is a real scalar function. In view of (34), we obtain

$$\frac{\mathrm{d}U_n(\mathbf{a})}{\mathrm{d}\lambda} = \nabla\mathbf{w}\cdot\frac{\mathrm{d}^n\mathbf{N}^a}{\mathrm{d}t^n},$$

where $\mathrm{d}^n\mathbf{N}^a/\mathrm{d}t^n$ is related by constitutive equation to $\mathbf{a}$ and $\nabla\mathbf{w}$ is gradient of $\mathbf{w}$. Making use of the above expression and (31), we arrive at

$$\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda} = \int\left\{\frac{\mathrm{d}^n\mathbf{N}^a}{\mathrm{d}t^n}\cdot(\nabla\mathbf{w})^{\mathrm{T}} - \frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n}\cdot\mathbf{w}\right\}\mathrm{d}\tau - \int_{S_{\mathrm{T}}}\left\{\frac{\mathrm{d}^n\mathbf{b}}{\mathrm{d}t^n} + \mathbf{R}_n + \mathbf{f}\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n} + \lambda\mathbf{w}\right)\right\}\cdot\mathbf{w}\,\mathrm{d}s. \tag{49}$$

Since $\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n$ is a true solution, by (39) we have

$$\int\left\{\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\cdot(\nabla\mathbf{w})^{\mathrm{T}} - \frac{\mathrm{d}^n\gamma}{\mathrm{d}t^n}\cdot\mathbf{w}\right\}\mathrm{d}\tau - \int_{S_{\mathrm{T}}}\left\{\frac{\mathrm{d}^n\mathbf{b}}{\mathrm{d}t^n} + \mathbf{R}_n + \mathbf{f}\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\right\}\cdot\mathbf{w}\,\mathrm{d}s = 0. \tag{50}$$

With (49) and (50), it is obvious that

$$\lambda\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda} = \int\left(\frac{\mathrm{d}^n\mathbf{N}^a}{\mathrm{d}t^n} - \frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right)\cdot\nabla\left(\mathbf{a} - \frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\mathrm{d}\tau - \int_{S_{\mathrm{T}}}\left\{\mathbf{f}\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n} + \lambda\mathbf{w}\right) - \mathbf{f}\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\right\}\cdot\left(\mathbf{a} - \frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\mathrm{d}s.$$

If $\lambda > 0$ and the sufficient uniqueness condition (48) holds, it follows that $\mathrm{d}g/\mathrm{d}\lambda > 0$. Since $g(0) = 0$, we conclude that $g(1) > 0$, i.e.

$$\prod_n\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}+\mathbf{w}\right)>\prod_n\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right).$$

Hence, when (48) is satisfied and the boundary condition on $S_T$ is self-adjoint, the value of functional $\prod_n$ reaches a strict absolute minimum at the true solution.

Return to the discussion of uniqueness. By the definitions of Green strain, the second Piola–Kirchhoff stress and nominal stress, we have

$$E=\tfrac{1}{2}(\mathbf{A}^{\mathrm{T}}\mathbf{A}-\mathbf{I}),\quad \mathbf{N}=\mathbf{T}\mathbf{A}^{\mathrm{T}},\tag{51}$$

where $\mathbf{I}$ is the unit tensor. Hence

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right)=\Delta\left(\frac{\mathrm{d}^n\mathbf{T}}{\mathrm{d}t^n}\right)\mathbf{A}^{\mathrm{T}}+\mathbf{T}\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right),\tag{52}$$

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)=\frac{1}{2}\left\{\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\mathbf{A}+\mathbf{A}^{\mathrm{T}}\Delta\left(\frac{\mathrm{d}^n\mathbf{A}}{\mathrm{d}t^n}\right)\right\}.\tag{53}$$

By making use of (52) and (53), the sufficient condition (48) can be written in terms of objective measures of stress and strain:

$$F_n=\int\left\{\Delta\left(\frac{\mathrm{d}^n\mathbf{T}}{\mathrm{d}t^n}\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)+\mathbf{T}\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\right\}\mathrm{d}\tau$$
$$-\int\mathbf{f}\left(\Delta\left(\frac{\mathrm{d}^n\mathbf{X}}{\mathrm{d}t^n}\right)\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{x}}{\mathrm{d}t^n}\right)\mathrm{d}s>0.\tag{54}$$

Because constitutive equations are non-linear, $F_n$ is not a single-valued function of $\Delta(\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n)$. Hence, direct application of the uniqueness test (48) or (54) is impossible. In the analysis of uniqueness of first-order rate problem, Hill (1958, 1959) constructed a linear comparison solid. Here we introduce a series of incremental linear comparison solids whose moduli $\mathbf{L}^n$ are defined by (6) with $\alpha$ determined as follows:

when $\mathbf{E}\in\Omega$, $\alpha=0$,

when $\mathbf{E}\in\partial\Omega$, for $n=1$, $\alpha=1$;

for $n=2$, $\alpha=0$ if $\mathbf{\Lambda}\cdot\dot{\mathbf{E}}<0$, $\alpha=1$ if $\mathbf{\Lambda}\cdot\dot{\mathbf{E}}\geqslant 0$;

for $n>2$, $\alpha=0$ if $\mathbf{\Lambda}\cdot\dot{\mathbf{E}}<0$ or $\mathbf{\Lambda}\cdot\dot{\mathbf{E}}=0$ and $\mathrm{d}^k/\mathrm{d}t^k\,(\mathbf{\Lambda}\cdot\dot{\mathbf{E}})<0$ for the smallest natural number $k\leqslant n-2$ which makes $\mathrm{d}^k/\mathrm{d}t^k\,(\mathbf{\Lambda}\cdot\dot{\mathbf{E}})\neq 0$, under other case $\alpha=1$. (55)

The following relative convexity relations can be easily proved:

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{T}}{\mathrm{d}t^n}\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)\geqslant\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)\cdot\mathbf{L}^n\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right),\tag{56}$$

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)\cdot\mathbf{L}^i\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)\geqslant\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right)\cdot\mathbf{L}^j\Delta\left(\frac{\mathrm{d}^n\mathbf{E}}{\mathrm{d}t^n}\right),\quad\text{for } 1\leqslant i<j\leqslant n.\tag{57}$$

For the particular case, $n=1$, the inequality (56) was given by Hill (1958). Let $n=2$, $j=2$ and $i=1$, then from (56) and (57) we get

$$\Delta\left(\frac{\mathrm{d}^2\mathbf{T}}{\mathrm{d}t^2}\right)\cdot\Delta\left(\frac{\mathrm{d}^2\mathbf{E}}{\mathrm{d}t^2}\right) \geqslant \Delta\left(\frac{\mathrm{d}^2\mathbf{E}}{\mathrm{d}t^2}\right)\cdot\mathbf{L}^1\Delta\left(\frac{\mathrm{d}^2\mathbf{E}}{\mathrm{d}t^2}\right).$$

The above inequality was presented by Petryk and Thermann (1985). With (52) and (53), the inequality (56) and (57) can be expressed in terms of nominal stress and deformation gradient:

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{N}}{\mathrm{d}t^n}\right)\cdot\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \geqslant \Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\cdot\mathbf{C}^n\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right), \tag{58}$$

$$\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\cdot\mathbf{C}^j\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right) \geqslant \Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right)\cdot\mathbf{C}^i\Delta\left(\frac{\mathrm{d}^n\mathbf{A}^{\mathrm{T}}}{\mathrm{d}t^n}\right), \quad 1 \leqslant i < j \leqslant n, \tag{59}$$

where $\mathbf{C}^i$ is related to $\mathbf{L}^i$ via (16) if $\mathbf{c}$ and $\mathbf{L}$ are replaced by $\mathbf{C}^i$ and $\mathbf{L}^i$ respectively. The following conditions are also sufficient for uniqueness of the solution to rate $n$ problem:

$$H_i = \int (\nabla\mathbf{w})^{\mathrm{T}}\cdot\mathbf{C}^i(\nabla\mathbf{w})^{\mathrm{T}}\,\mathrm{d}\tau - \int \mathbf{f}(\mathbf{w})\cdot\mathbf{w}\,\mathrm{d}s > 0, \quad 1 \leqslant i \leqslant n. \tag{60}$$

In (60), $\mathbf{w}$ is any continuous and piecewise smooth vector field vanishing on $S_{\mathrm{u}}$ and being not equal to zero. $H_1$ is Hill's exclusion functional. In fact, $\mathbf{w}$ can be regarded as $\Delta(\mathrm{d}^n\mathbf{x}/\mathrm{d}t^n)$. By (58) and (59), we have

$$F_n \geqslant H_i, \quad H_j \geqslant H_i, \quad 1 \leqslant i \leqslant j \leqslant n. \tag{61}$$

Hence, (60) implies (48). Now it is apparent that Hill's sufficient uniqueness condition is also sufficient for uniqueness of the solution to any order rate problem in the context of large strain, though it was first proposed by Hill for uniqueness of the solution to first-order rate problem. This conclusion was first indicated by Triantafyllidis (1983) for the case of the constitutive law proposed by Christofferson and Hutchinson (1979), by Petryk and Thermann (1985) for second-order rate problems and by Nguyen and Triantafyllidis (1989) for the case of small strain. All of them limited boundary conditions to that nominal tractions or displacements are prescribed on the body surface, that is, $\mathbf{f} = 0$.

## 5. BIFURCATION

In the section above, we have obtained the sufficient uniqueness conditions for rate $n$ problem. In this section, we will consider the problem of bifurcation. For this purpose we consider a quasistatic process of deformation, beginning where $H_1$ is positive definite. Suppose that a stage is reached, at which $H_1$ is positive semi-definite and vanishes for some non-zero continuous and piecewise continuously differentiable vector field $\mathbf{w}$ vanishing on $S_{\mathrm{u}}$. Then $\mathbf{w}$ is an eigenmode for the incremental linear comparison solid with moduli $\mathbf{L}^1$. If, at this moment, $\mathbf{L}^1$ coincides with the actual moduli tensor $\mathbf{L}^0$ in the considered process except possibly in a region of zero volume, the possible first and second order bifurcations have been analysed by Petryk and Thermann, and the higher order bifurcations can be analysed in similar ways. When $\mathbf{L}^1$ does not coincide with $\mathbf{L}^0$ and $\mathbf{w} \neq 0$ in a region of finite volume, $F_i > H_1 = 0$ and any bifurcation is impossible.

Suppose another stage in the considered process of deformation be reached, at which $H_2$ is positive semi-definite and vanishes for some non-zero admissible vector field $\mathbf{w}$ vanishing on $S_{\mathrm{u}}$. If now $\mathbf{L}^2$ does not coincide with $\mathbf{L}^0$ and $\mathbf{w} \neq 0$ in a region of finite volume, $F_i > H_2 = 0$ for $i \geqslant 2$. So we know that any order higher than one bifurcation is impossible. But when $\mathbf{L}^2 = \mathbf{L}^0$ except possibly in a region of zero volume, then various bifurcations are possible under some conditions. For example, the following particular cases can be distinguished:

(i) $\mathbf{E} \in \partial\Omega$ and $\mathbf{\Lambda} \cdot \dot{\mathbf{E}} = 0$ holds only in a material point set of zero volume. At this case $F_2 = H_2(\mathbf{w}) = 0$, so the second-order bifurcation is possible because $\mathbf{w}$ multiplied by any arbitrary coefficient $\mu$ can be added to the fundamental second-order solution to produce another second-order solution.

(ii) There is a region of finite volume in which $\mathbf{E} \in \partial\Omega$, $\mathbf{\Lambda} \cdot \dot{\mathbf{E}}_0 = 0$ and $\mathrm{d}/\mathrm{d}t\ (\mathbf{\Lambda} \cdot \dot{\mathbf{E}}_0) > c > 0$. Here $c$ is a constant and the script zero denotes quantities corresponding to the fundamental path. Then we can find a non-zero coefficient $\mu$ such that $(\mathrm{d}^2\mathbf{x}_0/\mathrm{d}t^2) + \mu\mathbf{w}$ is another second-order solution corresponding to the same moduli tensor $\mathbf{L}^0$ as the fundamental solution $\ddot{\mathbf{x}}_0$.

(iii) There is a region of finite volume in which $\mathbf{E} \in \partial\Omega$, $\mathbf{\Lambda} \cdot \dot{\mathbf{E}}_0 = 0$ and $\mathrm{d}/\mathrm{d}t\ (\mathbf{\Lambda} \cdot \dot{\mathbf{E}}_0) > 0$, but no such constant exists as in case (ii). Though the third-order bifurcation is possible, the second-order bifucation is impossible unless $\boldsymbol{\eta} \cdot (\nabla \mathbf{w})^{\mathrm{T}}$ is not changing sign over the region where $(\mathbf{\Lambda} \cdot \dot{\mathbf{E}}_0)^{\boldsymbol{\cdot}}$ tends to zero.

If the condition in case (i) is satisfied, the second-order rate problem becomes linear. The linear case has been discussed by Petryk and Thermann. However, in their final discussion they assumed that $\mathbf{L}^1 = \mathbf{L}^0$ except possibly in a region of zero volume. This is because the critical stage they examined is reached when $H_1 = 0$ and $F_i > H_1 = 0$ if $\mathbf{L}^1 \neq \mathbf{L}^0$ and $\mathbf{w} \neq 0$ in a region of finite volume. Here we examine the second-order bifurcation via $H_2 = 0$, so we have no need to assume that $\mathbf{L}^1 = \mathbf{L}^0$ nearly everywhere for the three cases listed above. Certainly, if $\mathbf{L}^1 \neq \mathbf{L}^0$ in a finite part, unloading (viz. $\mathbf{\Lambda} \cdot \dot{\mathbf{E}} < 0$) must occur momentarily in the same part and this perhaps does not take place frequently.

We can analyse other cases in similar ways. When $H_n = 0$ for any natural number $n$, the possible bifurcation can also be analysed. A simple model has been studied by Triantafyllidis which is capable of exhibiting a second-order bifurcation corresponding to case (i), the bifurcation of very high order perhaps is of no importance in practice. But to investigate rate boundary value problems of order higher than one perhaps is important for the analysis of initial post-bifurcation. To do so, the time-like parameter can not be chosen arbitrarily (cf. Hutchinson, 1973, 1974; Nguyen, 1987).

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